St George Girls' High School

Trial Higher School Certificate Examination

2000

Mathematics

4 Unit

Time Allowed: Three hours
(Plus 5 minutes reading time)

Directions to Candidates

- All questions may be attempted.
- All necessary working must be shown.
- Marks may be deducted for careless or poorly presented work.
- Begin each question on a new page.
- A list of standard integrals is included at the end of this paper.
- The mark allocated for each question is listed at the side of the question.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Question 1 – (15 Marks) **Marks**

a) For the two complex numbers

$$z_1 = (1+i)^2 \text{ and } z_2 = \sqrt{2}\left[\cos\left(\frac{^-5\pi}{6}\right) + i\sin\left(\frac{^-5\pi}{6}\right)\right]$$

(i) express z_1 in the modulus – argument form. 2

(ii) express in the $(a+ib)$ form, where a and b are real numbers:

z_2, $\bar{z}_2$, iz_1 3

(iii) find (x, y) such that $(z_2)^6 = x + iy$ 2

b) Determine the complex square roots of $2 - 2\sqrt{3}i$. Express your answer in the form $a + ib$.

Diff relates polynom

c) Find, in simplest form, the quadratic equation whose roots are $2-i$ and $(2-i)^{-1}$ 4

Question 2 – (15 Marks) **Marks**

a) Find the indefinite integrals

(i) $\int \frac{2x}{(x+4)(x+3)}dx$ **2**

(ii) $\int x^3 e^{x^4+1} dx$ **1**

(iii) $\int \frac{dx}{\sqrt{8x-4x^2}}$ **2**

b) Evaluate

(i) $\int_0^{\frac{\pi}{2}} \sin^2 x \quad \cos^3 x \quad dx$ **3**

(ii) Show that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, and, hence, show that

$$\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\sin^4 x + \cos^4 x} dx$$ **3**

Hence, evaluate each integral

c) (i) If $U_n = \int_0^{\frac{\pi}{2}} \cos^n \theta \quad d\theta$, prove that $U_n = \frac{n-1}{n} U_{n-2}$, for $n \geq 2$ **3**

(ii) Hence, evaluate $\int_0^{\frac{\pi}{2}} \cos^9 \theta \quad d\theta$ **1**

Question 3 – (15 Marks) **Marks**

a) If one of the roots of $P(x) = x^3 - x^2 - 6x + 18 = 0$ is $2 - \sqrt{2}i$, find the other two roots. 3

b) If α, β and γ are the roots of the equation $x^3 - 3x^2 + 6x + 7 = 0$, find

(i) the polynomial equation whose roots are $\alpha^2, \beta^2, \gamma^2$ 3

(ii) $\alpha^3 + \beta^3 + \gamma^3$ 3

c) The roots of the equation $t^3 + qt - r = 0$ are a, b and c. If $S_n = a^n + b^n + c^n$, where n is a positive integer, prove that: $S_{n+3} = rS_n - qS_{n+1}$ 3

d) The region bounded by $y = x^4$, $0 \le x \le 2$, and the x – axis is rotated about the line $x = 3$. Use the method of "cylindrical shells" to find the volume generated. (Leave the answer in terms of π).

Question 4 – (15 Marks) **Marks**

a) If β is a complex root of $z^5 = 1$. **5**

(i) Show that the roots are of the form $1, \beta, \beta^2, \beta^3, \beta^4$.

(ii) Find the value of $1 + \beta + \beta^2 + \beta^3 + \beta^4$.

(iii) Show that $\beta^{-1} = \beta^4$ and $\beta^{-2} = \beta^3$.

(iv) Hence find the quadratic equation with roots of $\beta + \beta^{-1}$ and $\beta^2 + \beta^{-2}$.

(v) Deduce that

$$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$$ **2**

b) What is the maximum value of $|z|$ if $|z + 1 + 2i| \leq 1$? **3**

c) If $|z_1 - z_2| = |z_1 + z_2|$, show that $\arg \ z_2 - \arg \ z_1 = \frac{\pi}{2}$. **2**

d) Evaluate $\int_0^{\frac{2}{3}} \sqrt{4 - 9u^2}\ du$ **3**

Question 5 – (15 Marks) **Marks**

a) Find the limiting sum of the series $\frac{1}{5}+\frac{2}{5^2}+\frac{3}{5^3}+\ldots\ldots$ **3**

b) P and Q are the points t_1 and t_2 on the rectangular hyperbola $xy=c^2$.

(i) Show that the gradient of PQ is $\frac{-1}{t_1t_2}$. **1**

(ii) Hence, or otherwise, prove that if PQ subtends a right angle at a third point R on the hyperbola, then the tangent at R is perpendicular to PQ. **4**

c) (i) By expanding $\cos(2\theta+\theta)$, show that $\cos^3\theta-\frac{3}{4}\cos\theta=\frac{1}{4}\cos 3\theta$. **2**

(ii) By making the substitution $x=m\cos\theta$ for a suitable value of m, and using the result in (i), find the roots of:

$$27x^3-9x-\sqrt{3}=0$$ **4**

Express them in the form $m\cos\theta$.

(iii) Hence, prove that

$$\cos\frac{\pi}{18}.\cos\frac{3\pi}{18}.\cos\frac{5\pi}{18}.\cos\frac{7\pi}{18}=\frac{3}{16}$$ **1**

Question 6 – (15 Marks) **Marks**

The hyperbola H has Cartesian equation $\frac{x^2}{4} - \frac{y^2}{7} = 1$.

a) Write down its eccentricity, the coordinates of its foci S and S^1, the equations of the directrices and the equations of the asymptotes. **4**

b) Sketch the curve and include all details of part (a). **2**

c) P is an arbitrary point $(2\sec\theta, \sqrt{7}\tan\theta)$. Show that P lies on H and prove that the tangent to H at P has equation $\frac{x\sec\theta}{2} - \frac{y\tan\theta}{\sqrt{7}} = 1$. **4**

d) This tangent cuts the asymptotes in L and M.

α) Prove that $LP = PM$ and **2**

β) Prove the area of ΔOLM is independent of the position of P on H. (O is the origin). **3**

Question 7 – (15 Marks) **Marks**

a) Sketch the following graphs, indicating their essential features.
(Draw on separate graphs).

(i) $y = \sqrt{\ln x}$ **2**

(ii) $y = \tan^{-1}(\tan x)$ for $\frac{-3\pi}{2} \le x \le \frac{3\pi}{2}$ **2**

(iii) $|y| = \sin x$ for $0 \le x \le 3\pi$ **2**

(iv) $y = \frac{x-1}{x^2 - 4}$ **3**

Not covered yet

b) The base of a solid is the circle $x^2 + y^2 = 6x$. Every plane section perpendicular to the x – axis is a rectangle whose height is one half of the distance of the plane section from the origin. Find the volume of the solid. **6**

Question 8 – (cont'd) **Marks**

(i) Show that the projectile is above the x – axis for a total of $\dfrac{2V\sin\alpha}{g}$ seconds. **1**

(ii) Show that the horizontal range of the projectile is $\dfrac{2V^2\sin\alpha\cos\alpha}{g}$ metres. **1**

(iii) At the instant the projectile is fired, the target T is d metres from O and it is moving away at a constant speed of u *m/s*.

Suppose that the projectile hits the target when fired at an angle of elevation α.

Show that:

$$u = V\cos\alpha - \frac{gd}{2V\sin\alpha}$$ **3**

In parts (iv) and (v), assume that $gd = \dfrac{V^2}{2\sqrt{3}}$.

(iv) By using (iii) and the graph of part (a), show that if $u > \dfrac{V}{\sqrt{3}}$ the target cannot be hit by the projectile, no matter at what angle of elevation α the projectile is fired. **3**

(v) Suppose $u < \dfrac{V}{\sqrt{3}}$. Show that the target can be hit when it is at precisely two distances from O. **2**

The following list of standard integrals may be used:

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq 1; \; x \neq 0, \; \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right).$$

Note: $\ln x = \log_e x, \quad x > 0$

TRIAL HSC
4 UNIT COURSE
SOLUTIONS.

QUESTION 1:

(a) (i) $z_1 = (1+i)^2$
$= 1 + 2i + i^2$
$= 2i$
$= 2\,\text{cis}\,\frac{\pi}{2}.$

(ii) $z_2 = \sqrt{2}\left[-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right]$
$= -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$
$\bar{z}_2 = -\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$
$iz_1 = i \cdot 2i$
$= -2$

(iii) $z_2 = \sqrt{2}\left[\text{cis}\left(-\frac{5\pi}{6}\right)\right]$

$z_2^6 = 8\left[\text{cis}\left(-5\pi\right)\right]$
$= 8\,\text{cis}\,\pi$
$= -8.$

(b) let $a + ib = \sqrt{2 - 2\sqrt{3}i}$

squaring
$\Rightarrow a^2 - b^2 + 2abi = 2 - 2\sqrt{3}i$
$\Rightarrow a^2 - b^2 = 2$ ——— (1)
$ab = -\sqrt{3}$ ——— (2)

from (2) $b = -\frac{\sqrt{3}}{a}$ sub in (1)

$\therefore a^2 - \frac{3}{a^2} = 2$

$a^4 - 2a^2 - 3 = 0$
$(a^2 - 3)(a^2 + 1) = 0$
$a^2 = 3$
$a = \sqrt{3}, -\sqrt{3}$
$b = -1, 1$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^4 x + \cos^4 x}{\sin^4 x + \cos^4 x}\, dx$$

$$= \int_0^{\frac{\pi}{2}} 1\, dx$$

$$= \frac{\pi}{2}$$

and since the two integrals are of equal value, each integrals is equal to $\frac{\pi}{4}$.

(c) (i) $$U_n = \int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta$$

$$= \int_0^{\pi/2} \underbrace{\cos^{n-1}\theta}_{v} . \underbrace{\cos\theta \, d\theta}_{du}$$

$$= \left[\sin\theta . \cos^{n-1}\theta\right]_0^{\frac{\pi}{2}} - \int_0^{\pi/2} \sin\theta . (n-1)\cos^{n-2}\theta . -\sin \ldots$$

$$= (n-1) \int_0^{\pi/2} \sin^2\theta . \cos^{n-2}\theta \, d\theta$$

$$= (n-1) \int_0^{\frac{\pi}{2}} (1-\cos^2\theta) . \cos^{n-2}\theta \, d\theta$$

$$= (n-1)\left[U_{n-2} - U_n\right]$$

$$U_n = (n-1)U_{n-2} - (n-1)U_n$$

$$U_n(1+n-1) = (n-1)U_{n-2}$$

$$\therefore U_n = \left(\frac{n-1}{n}\right) U_{n-2} \qquad n \geq 2$$

(ii) $$\int_0^{\pi/2} \cos^9\theta \, d\theta = U_9$$

$$= \frac{8}{9} \times U_7$$

$$= \frac{8}{9} \times \frac{6}{7} \times U_5$$

$$= \frac{8}{9} \times \frac{6}{7} \times \frac{4}{5} \times U_3$$

$$= \frac{8}{9} \times \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} U_1$$

$$= \frac{128}{315} \times \int_0^{\pi/2} \cos\theta \, d\theta$$

$$= \frac{128}{315} \times \left[\sin\theta\right]_0^{\pi/2}$$

$$= \frac{128}{315}$$

(c) Sum of roots $= 2-i + \frac{1}{2-i}$

$= 2-i + \frac{2+i}{5}$

$= \frac{12}{5} - \frac{4}{5}i$

Product of roots $= 2-i \times \frac{1}{2-i}$

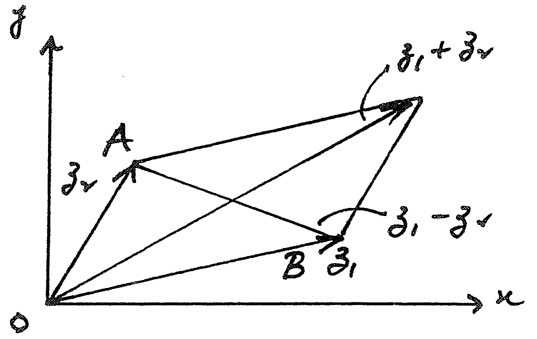
$= 1$

$\therefore$ Quadratic equation is

$x^2 - \left(\frac{12}{5} - \frac{4}{5}i\right) + 1 = 0$

ie $5x^2 - (12-4i) + 5 = 0$

(d)



$|z_1 + z_2| = |z_1 - z_2| \Rightarrow$ diagonals of parallelogram are equal

$\Rightarrow$ ∥ogram is a square or rectangle

$\Rightarrow A\hat{O}B = \frac{\pi}{2}$

$\therefore \arg z_2 - \arg z_1 = \frac{\pi}{2}$

2

(e) $\int_0^{\frac{2}{3}} \sqrt{4 - 9u^2}\, du$

let $3u = 2\sin\theta$
$3\,du = 2\cos\theta\, d\theta$

$= \int_0^{\frac{\pi}{2}} \sqrt{4 - 4\sin^2\theta} \cdot \frac{2\cos\theta}{3}\, d\theta$

$= \frac{4}{3} \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$= \frac{4}{3} \cdot \frac{1}{2} \cdot \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)\, d\theta$

$= \frac{2}{3} \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi/2}$

$= \frac{2}{3} \left[\frac{\pi}{2} - 0 \right]$

$= \frac{\pi}{3}$

3

or $I = 3\int_0^{2/3} \sqrt{\left(\frac{2}{3}\right)^2 - u^2}\, du$

$= 3 \times \frac{1}{4} \pi r^2$

$= \frac{3}{4} \cdot \pi \cdot \frac{4}{9}$

$= \frac{\pi}{3}$

$x^2 + y^2 = \left(\frac{2}{3}\right)^2$

$y = \sqrt{\frac{4}{9} - x^2}$

QUESTION 2

(a) (i) $\displaystyle\int \frac{2x}{(x+4)(x+3)}\,dx$

let $\dfrac{2x}{(x+4)(x+3)} = \dfrac{a}{x+4} + \dfrac{b}{x+3}$

ie $2x = a(x+3) + b(x+4)$

$x=-3 \Rightarrow -6 = b$

$x=-4 \Rightarrow -8 = -a$

$\therefore a = 8$

$\therefore \dfrac{2x}{(x+4)(x+3)} = \dfrac{8}{x+4} - \dfrac{6}{x+3}$

$\therefore \displaystyle\int \frac{2x}{(x+4)(x+3)}\,dx = 8\ln|x+4| - 6\ln|x+3| + c$

(ii) $\displaystyle\int x^3 e^{x^4+1}\,dx$ $\qquad u = e^{x^4+1}$

$du = 4x^3 e^{x^4+1}\,dx$

$= \frac{1}{4}\displaystyle\int 4x^3 e^{x^4+1}\,dx$

$= \frac{1}{4}\displaystyle\int e^u\,du$

$= \frac{1}{4}\cdot e^u + c$

$= \frac{1}{4}\cdot e^{x^4+1} + c$

(iii) $8x - 4x^2 = -4(x^2-2x)$

$= -4[(x^2-2x+1)-1]$

$= 4 - 4(x-1)^2$

$\displaystyle\int \frac{dx}{\sqrt{8x-4x^2}} = \int \frac{dx}{2\sqrt{1-(x-1)^2}}$

$= \frac{1}{2}\sin^{-1}(x-1) + c$

QUESTION 3:

(a) $P(x) = x^3 - x^2 - 6x + 18$

since co-effs of $P(x)$ are real

$x = 2 - \sqrt{2}i$ a zero $\Rightarrow$ $= 2 + \sqrt{2}i$ also a zero

$\therefore$ $x^2 - [2 - \sqrt{2}i + 2 + \sqrt{2}i]x + (2-\sqrt{2}i)(2+\sqrt{2}i)$ is a factor of $P(x)$

ie $x^2 - 4x + 6$ is a factor

$\therefore$ $P(x) = (x^2 - 4x + 6)(x + 3)$

$\therefore$ Other roots are $2 + \sqrt{2}i$, -3

(b) $P(x) = x^3 - 3x^2 + 6x + 7$

(i) $y = x^2 \Rightarrow x = \sqrt{y}$

$\therefore$ $P(\sqrt{x}) = 0$

$$\Rightarrow (\sqrt{x})^3 - 3(\sqrt{x})^2 + 6\sqrt{x} + 7 = 0$$

$$x\sqrt{x} - 3x + 6\sqrt{x} + 7 = 0$$

$$\sqrt{x}(x+6) = 3x - 7$$

$$x(x+6)^2 = (3x-7)^2$$

$$x^3 + 12x^2 + 36x = 9x^2 - 42x + 49$$

ie $x^3 + 3x^2 + 78x - 49 = 0$ ——— (1)

(ii) since α, β, γ are roots of $P(x) = 0$

$$\alpha^3 - 3\alpha^2 + 6\alpha + 7 = 0$$

$$\beta^3 - 3\beta^2 + 6\beta + 7 = 0$$

$$\gamma^3 - 3\gamma^2 + 6\gamma + 7 = 0$$

adding $\Rightarrow$ $\Sigma\alpha^3 - 3\Sigma\alpha^2 + 6\Sigma\alpha + 21 = 0$

$\Sigma\alpha^3 = 3(-3) - 6(3) - 21$ $\quad$ $\Sigma\alpha^2$ from (1)

$= -48$

(c) $t^3 + qt - r = 0$

a, b, c roots

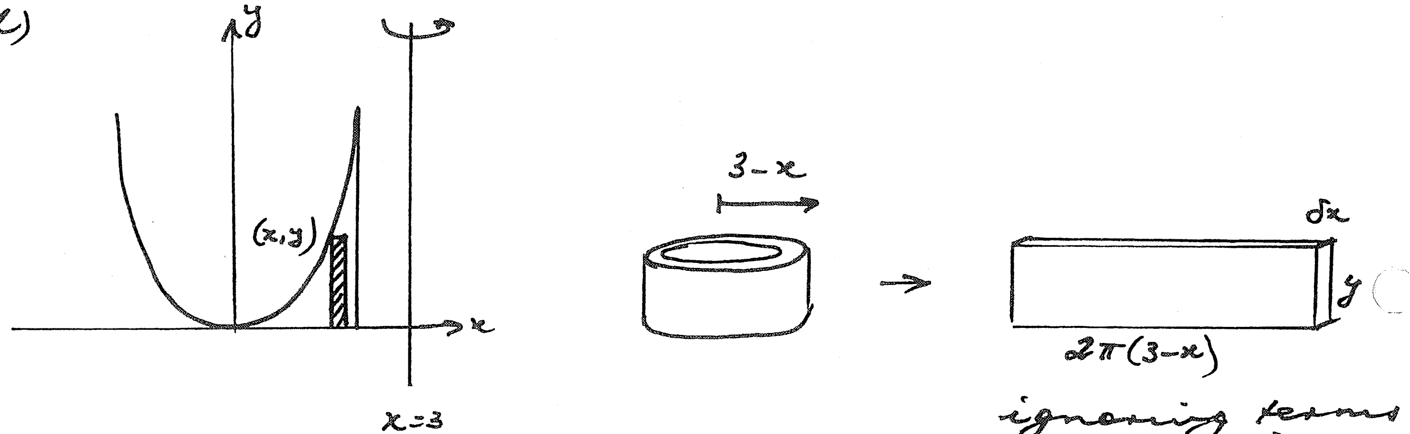
$$\Rightarrow \quad a^3 + qa - r = 0$$
$$b^3 + qb - r = 0$$
$$c^3 + qc - r = 0$$

adding $\Rightarrow$ $a^3 + b^3 + c^3 = -q(a+b+c) + 3r$
$= 3r$

Then $S_{n+3} = a^{n+3} + b^{n+3} + c^{n+3}$

$$= a^n . a^3 + b^n . b^3 + c^n . c^3$$
$$= a^n(r - qa) + b^n(r - qb) + c^n(r - qc)$$
$$= r(a^n + b^n + c^n) - q(a^{n+1} + b^{n+1} + c^{n+1})$$
$$= r S_n - q S_{n+1}$$

(d)



ignoring terms in $(\delta x)^2$

volume of shell is $\delta V = 2\pi(3-x)y\,\delta x$

$\therefore$ Volume of solid is $V = \lim_{\delta x \to 0} \sum_{x=0}^{2} 2\pi(3-x)\,y\,\delta x$

$$= 2\pi \int_0^2 (3-x).\,x^4\,dx$$
$$= 2\pi \left[\frac{3x^5}{5} - \frac{x^6}{6}\right]_0^2$$
$$= 2\pi \left[\frac{96}{5} - \frac{64}{6} - 0\right]$$
$$= \frac{256\pi}{15} \text{ units}^3$$

QUESTION 4:

(a) $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$

Then $\dfrac{a + b\omega + c\omega^2}{c + a\omega + b\omega^2} = \dfrac{a + \frac{b}{\omega^2} + \frac{c}{\omega}}{c + \frac{a}{\omega^2} + \frac{b}{\omega}}$

$= \dfrac{a\omega^2 + b + c\omega}{c\omega^2 + a + b\omega}$

$= \dfrac{\omega^2(a + b\omega + c\omega^2)}{a + b\omega + c\omega^2}$ since $\omega^3 = 1$

$= \omega^2$

(b)(i) $z^5 = 1$

$= 1\,\text{cis}\,(0 + 2k\pi)$ k integer

$\therefore z = 1\,\text{cis}\,\frac{2k\pi}{5}$

$k=0 \Rightarrow z = 1$

$k=1 \Rightarrow z = \text{cis}\,\frac{2\pi}{5} = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$k=2 \Rightarrow z = \text{cis}\,\frac{4\pi}{5} = -\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$

$k=3 \Rightarrow z = \text{cis}\,\frac{6\pi}{5} = -\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}$ (2)

$k=4 \Rightarrow z = \text{cis}\,\frac{8\pi}{5} = \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$

(ii) If $\beta = \text{cis}\,\frac{2\pi}{5}$, $\beta^{-1} = \text{cis}\left(-\frac{2\pi}{5}\right)$ (1)

$= \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5} = \beta^4$

$\beta + \beta^{-1} = 2\cos\frac{2\pi}{5}$ (1)

and $\beta^2 = \text{cis}\,\frac{4\pi}{5} = -\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$\beta^{-2} = \text{cis}\left(-\frac{4\pi}{5}\right) = -\cos\frac{\pi}{5} - i\sin\frac{4\pi}{5}$

$$\therefore \quad \beta^2 + \beta^{-2} = -2\cos\frac{\pi}{5} \quad \text{———} \; (2)$$

$\therefore$ Quadratic equation

is $$z^2 - \left(2\cos\frac{2\pi}{5} - 2\cos\frac{\pi}{5}\right)z - 4\cos\frac{\pi}{5}\cos\frac{2\pi}{5} = 0 \quad \text{———} \; (3)$$

now $\beta^5 = 1$

$$\Rightarrow \beta^5 - 1 = 0$$

$$(\beta - 1)(\beta^4 + \beta^3 + \beta^2 + \beta + 1) = 0 \qquad (1)$$

$$\beta \neq 1 \Rightarrow \beta^4 + \beta^3 + \beta^2 + \beta + 1 = 0$$

$$\therefore \beta + \beta^2 + \beta^3 + \beta^4 = -1$$

If roots are $\beta + \beta^{-1}$, $\beta^2 + \beta^{-2}$ then quadratic is

$$z^2 - (\beta + \beta^{-1} + \beta^2 + \beta^{-2})z + (\beta + \beta^{-1})(\beta^2 + \beta^{-2}) = 0$$

$$z^2 - (\beta + \beta^4 + \beta^2 + \beta^3)z + (\beta^3 + \beta^4 + \beta + \beta^2) = 0 \qquad (1)$$

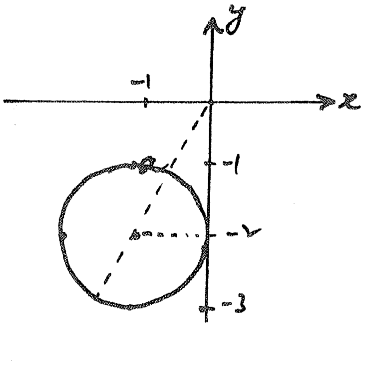
$$\text{ie } z^2 + z - 1 = 0 \quad \text{———} \; (4)$$

Comparing (3) and (4)

$$\Longrightarrow 2\cos\frac{2\pi}{5} - 2\cos\frac{\pi}{5} = -1 \qquad (2)$$

$$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2} \qquad \text{QED.}$$

(c) b) $|z + 1 + 2i| \leq 1$



$$a^2 = 2^2 + 1^2$$
$$\therefore a = \sqrt{5}$$

maximum value of $|z|$ is $\sqrt{5} + 1$ (3)

QUESTION 5:

(a) $\frac{1}{5}+\frac{2}{5^2}+\frac{3}{5^3}$

$= \frac{1}{5}+\left(\frac{1}{5^2}+\frac{1}{5^2}\right)+\left(\frac{1}{5^3}+\frac{1}{5^3}+\frac{1}{5^3}\right)+\left(\frac{1}{5^4}+\frac{1}{5^4}+\frac{1}{5^4}+\frac{1}{5^4}\right)$

$= \left(\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\cdots\right)+\left(\frac{1}{5^2}+\frac{1}{5^3}+\frac{1}{5^4}+\cdots\right)$

$+\left(\frac{1}{5^3}+\frac{1}{5^4}+\frac{1}{5^5}+\cdots\right)+\left(\frac{1}{5^4}+\frac{1}{5^5}+\cdots\right)+\cdots$

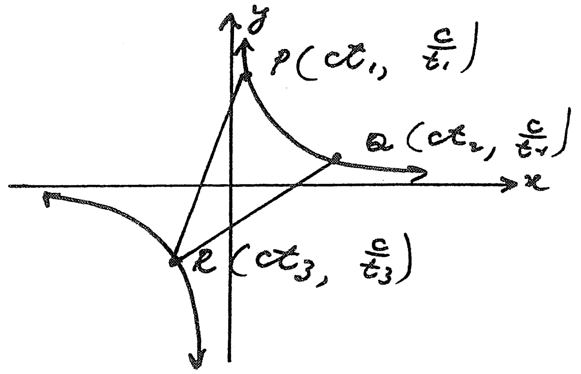
$= \frac{\frac{1}{5}}{\frac{4}{5}}+\frac{\frac{1}{5^2}}{\frac{4}{5}}+\frac{\frac{1}{5^3}}{\frac{4}{5}}+\cdots$

$= \frac{1}{4}+\frac{1}{20}+\frac{1}{100}+\cdots$ the common ratio clearly being $\frac{1}{5}$

$= \frac{\frac{1}{4}}{\frac{4}{5}}$

$= \frac{5}{16}$

(b)



(i) $m_{PQ} = \frac{\frac{c}{t_2}-\frac{c}{t_1}}{ct_2-ct_1} \times \left(\frac{t_1t_2}{t_1t_2}\right)$

$= \frac{c(t_1-t_2)}{ct_1t_2(t_2-t_1)}$

$= -\frac{1}{t_1t_2}$

(ii) $PR \perp QR \Rightarrow -\frac{1}{t_1t_3} \times \frac{-1}{t_2t_3} = -1$

$\therefore t_1t_2t_3^2 = -1$ —— (1)

now $xy = c^2$

$\Rightarrow y = c^2 x^{-1}$

$y' = -c^2 x^{-2}$

at $R\left(ct_3, \frac{c}{t_3}\right)$ $y' = \dfrac{-c^2}{c^2 t_3^2}$

ie $m_R = -\dfrac{1}{t_3^2}$ —— $\cancel{2}$ 1

Then $m_R \times m_{PQ} = -\dfrac{1}{t_3^2} \times \dfrac{-1}{t_1 t_2}$

$= \dfrac{1}{t_1 t_2 t_3^2}$

$= -1$ from ① —— 2

$\therefore$ Tangent at R must be $\perp$ PQ.

(c) (i) $\cos 3\theta = \cos(2\theta + \theta)$

$= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$

$= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta . \sin\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta(1 - \cos^2\theta)$

$= 4\cos^3\theta - 3\cos\theta$

$\Rightarrow \cos^3\theta - \frac{3}{4}\cos\theta = \frac{1}{4}\cos 3\theta$ —— ①

(ii) let $x = m\cos\theta$ in $27x^3 - 9x - \sqrt{3} = 0$

ie $27m^3\cos^3\theta - 9m\cos\theta - \sqrt{3} = 0$ —— E

$\therefore \cos^3\theta - \dfrac{1}{3m^2}\cos\theta = \dfrac{\sqrt{3}}{27m^3}$ —— ②

Comparing with ①, $\dfrac{1}{3m^2} = \dfrac{3}{4}$

ie $\quad m^2 = \frac{4}{9}$

$\therefore m = \frac{2}{3}$ will suffice

(2) then becomes

$$\cos^3\theta - \frac{3}{4}\cos\theta = \frac{\sqrt{3}}{27 \times \frac{8}{27}}$$

$$= \frac{\sqrt{3}}{8}$$

$$\Longrightarrow \quad \frac{1}{4}\cos 3\theta = \frac{\sqrt{3}}{8}$$

$$\therefore \cos 3\theta = \frac{\sqrt{3}}{2}$$

$$3\theta = 2n\pi \pm \frac{\pi}{6}$$

$$\therefore 3\theta = \pm\frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}, \frac{25\pi}{6}, \dots$$

$$\text{ie } \theta = \pm\frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{23\pi}{18}, \frac{25\pi}{18}, \dots$$

The 3 solutions of

$27x^3 - 9x - \sqrt{3} = 0$ will be

$$\frac{2}{3}\cos\frac{\pi}{18}, \frac{2}{3}\cos\frac{11\pi}{18}, \frac{2}{3}\cos\frac{13\pi}{18}, \frac{2}{3}\cos\frac{23\pi}{18}, \frac{2}{3}\cos\frac{25\pi}{18}, \dots$$

$$= \frac{2}{3}\cos\frac{\pi}{18}, -\frac{2}{3}\cos\frac{7\pi}{18}, -\frac{2}{3}\cos\frac{5\pi}{18}, -\frac{2}{3}\cos\frac{5\pi}{18}, -\frac{2}{3}\cos\frac{7\pi}{18}, \dots$$

$$\text{ie } \frac{2}{3}\cos\frac{\pi}{18}, -\frac{2}{3}\cos\frac{7\pi}{18}, -\frac{2}{3}\cos\frac{5\pi}{18}.$$

Product of roots

$$\Longrightarrow \frac{2}{3}\cos\frac{\pi}{18} \cdot -\frac{2}{3}\cos\frac{7\pi}{18} \cdot -\frac{2}{3}\cos\frac{5\pi}{18} = \frac{\sqrt{3}}{27}$$

$$\text{ie } \frac{8}{27}\cos\frac{\pi}{18} \cdot \cos\frac{5\pi}{18} \cdot \cos\frac{7\pi}{18} = \frac{\sqrt{3}}{27}$$

$$\cos\frac{\pi}{18} \cdot \cos\frac{5\pi}{18} \cdot \cos\frac{7\pi}{18} = \frac{\sqrt{3}}{}$$

(b) (i) $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x \, dx$

$= \int_0^{\frac{\pi}{2}} \sin^2 x \cdot \cos^2 x \cdot \cos x \, dx$ $\qquad u = \sin x$

$du = \cos x \, dx$

$= \int_0^1 u^2(1-u^2)\, du$

$= \int_0^1 (u^2 - u^4)\, du$

$= \left[\frac{u^3}{3} - \frac{u^5}{5}\right]_0^1$

$= \frac{1}{3} - \frac{1}{5}$

$= \frac{2}{15}.$

(ii) Consider $\int_0^a f(a-x)\, dx$ $\qquad$ let $u = a - x$

$du = -dx$

$= \int_a^0 f(u) \cdot -du$

$= \int_0^a f(u)\, du$

$= \int_0^a f(x)\, dx$

$$\int_0^{\pi/2} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\, dx = \int_0^{\pi/2} \frac{\sin^4(\frac{\pi}{2} - x)}{\sin^4(\frac{\pi}{2}-x) + \cos^4(\frac{\pi}{2}-x)}$$

$$= \int_0^{\pi/2} \frac{\cos^4 x}{\cos^4 x + \sin^4 x}\, dx$$

Then $\int_0^{\pi/2} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\, dx + \int_0^{\pi/2} \frac{\cos^4 x}{\sin^4 x + \cos^4 x}\, dx$

$$\therefore \cos\frac{\pi}{18}\cdot\cos\frac{3\pi}{18}\cdot\cos\frac{5\pi}{18}\cdot\cos\frac{7\pi}{18} = \frac{\sqrt{3}}{8}\cdot\cos\frac{3\pi}{18}$$

$$= \frac{\sqrt{3}}{8}\cdot\frac{\sqrt{3}}{2}$$

$$= \frac{3}{16}$$

QUESTION 6:

$$\frac{x^2}{4} - \frac{y^2}{7} = 1 \quad \Rightarrow \quad a = 2$$
$$b = \sqrt{7}$$

(a)

$$b^2 = a^2(e^2 - 1)$$
$$7 = 4(e^2 - 1)$$
$$e^2 - 1 = \frac{7}{4}$$
$$e^2 = \frac{11}{4}$$
$$\therefore e = \frac{\sqrt{11}}{2} \quad \text{—— } 1$$

$$e = \frac{\sqrt{11}}{2}$$

Foci at $(\pm ae, 0)$ ie $(\pm \sqrt{11}, 0)$ —— 1

$\therefore$ Directrices are $x = \pm \frac{a}{e}$

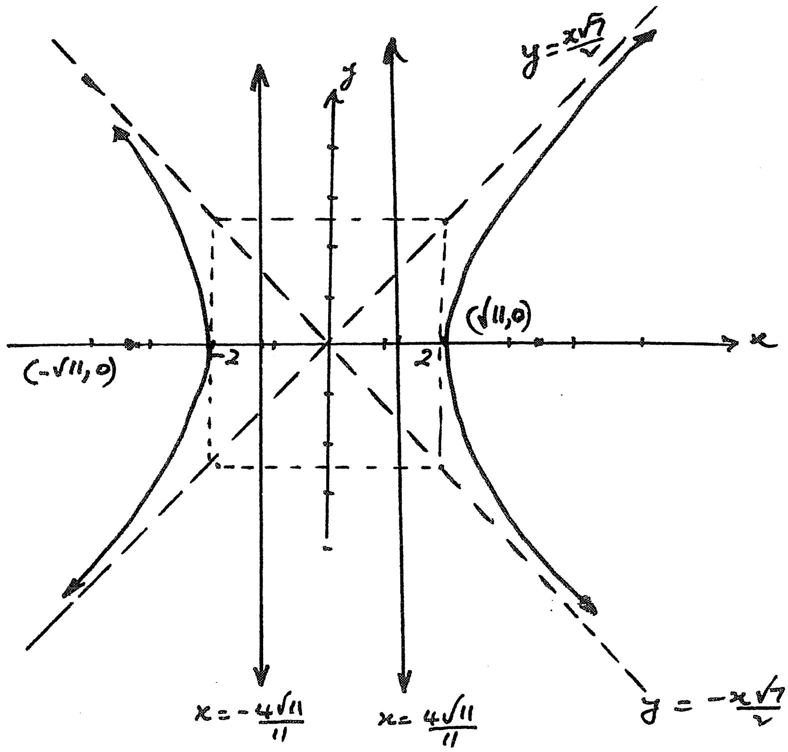
$$= \pm \frac{2}{\left(\frac{\sqrt{11}}{2}\right)} \quad \text{—— } 1$$

$$= \pm \frac{4}{\sqrt{11}} \quad \text{OR} \quad \pm \frac{4\sqrt{11}}{11}$$

Asymptotes are $y = \pm \frac{bx}{a}$

$$= \pm \frac{\sqrt{7}\,x}{2} \quad \text{—— } 1$$

(b)



(c) P is $(2\sec\theta, \sqrt{7}\tan\theta)$

Then $\frac{x^2}{4} - \frac{y^2}{7} = \frac{4\sec^2\theta}{4} - \frac{7\tan^2\theta}{7}$

$= \sec^2\theta - \tan^2\theta$

$= 1$

$\therefore$ P lies on H.

$$\frac{x^2}{4} - \frac{y^2}{7} = 1$$

differentiating w.r.t. x

$\Longrightarrow \quad \frac{x}{2} - \frac{2yy'}{7} = 0$

ie $\frac{2yy'}{7} = \frac{x}{2}$

$y' = \frac{7x}{4y}$

at P: $\quad y' = \frac{14\sec\theta}{4\sqrt{7}\tan\theta}$

$= \frac{\sqrt{7}\sec\theta}{2\tan\theta}$

Tangent at P is

$$y - \sqrt{7}\tan\theta = \frac{\sqrt{7}\sec\theta}{2\tan\theta}(x - 2\sec\theta)$$

ie $2\tan\theta . y - 2\sqrt{7}\tan^2\theta = \sqrt{7}\sec\theta . x - 2\sqrt{7}\sec^2\theta$

ie $x\sqrt{7}\sec\theta - 2y\tan\theta = 2\sqrt{7}(\sec^2\theta - \tan^2\theta)$

$= 2\sqrt{7}$

$$\therefore \frac{x\sec\theta}{2} - \frac{y\tan\theta}{\sqrt{7}} = 1 \quad \text{——— (1)}$$

(i) solving (1) with $y = \frac{x\sqrt{7}}{2}$

$$\Rightarrow \frac{x\sec\theta}{2} - \frac{x\sqrt{7}}{2\sqrt{7}}\tan\theta = 1$$

$$x\sqrt{7}(\sec\theta - \tan\theta) = 2\sqrt{7}$$

$$\Rightarrow x = \frac{2}{\sec\theta - \tan\theta}$$

$$= \frac{2(\sec\theta + \tan\theta)}{\sec^2\theta - \tan^2\theta}$$

$$= 2(\sec\theta + \tan\theta)$$

ie L is $\left(2(\sec\theta + \tan\theta), \sqrt{7}(\sec\theta + \tan\theta)\right)$

Solving (1) with $y = -\frac{x\sqrt{7}}{2}$

$$\Rightarrow \frac{x\sec\theta}{2} + \frac{x\sqrt{7}}{2\sqrt{7}}\tan\theta = 1$$

$$x\sqrt{7}(\sec\theta + \tan\theta) = 2\sqrt{7}$$

$$\therefore x = \frac{2}{\sec\theta + \tan\theta}$$

$$= 2(\sec\theta - \tan\theta)$$

$\therefore$ M is $\left(2(\sec\theta - \tan\theta), -\sqrt{7}(\sec\theta - \tan\theta)\right)$

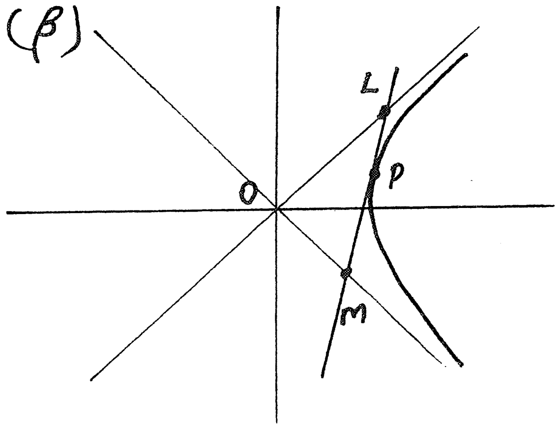
(α) Then the mid-point of LM is

$(2\sec\theta,\ \sqrt{7}\tan\theta)$

which is P

$\therefore\ LP = MP$

(β)



Equation LO is $y = \frac{x\sqrt{7}}{2}$ ie $x\sqrt{7} - 2y = 0$

$\perp$ distance from M to LO is

$$d = \frac{\left|2\sqrt{7}(\sec\theta - \tan\theta) + 2\sqrt{7}(\sec\theta - \tan\theta)\right|}{\sqrt{11}}$$

$$= \frac{\left|4\sqrt{7}(\sec\theta - \tan\theta)\right|}{\sqrt{11}}$$

Distance $OL = \sqrt{4(\sec\theta + \tan\theta)^2 + 7(\sec\theta + \tan\theta)^2}$

$$= \sqrt{11\sec^2\theta + 22\sec\theta\tan\theta + 11\tan^2\theta}$$

$$= \sqrt{11}\,\left|\sec\theta + \tan\theta\right|$$

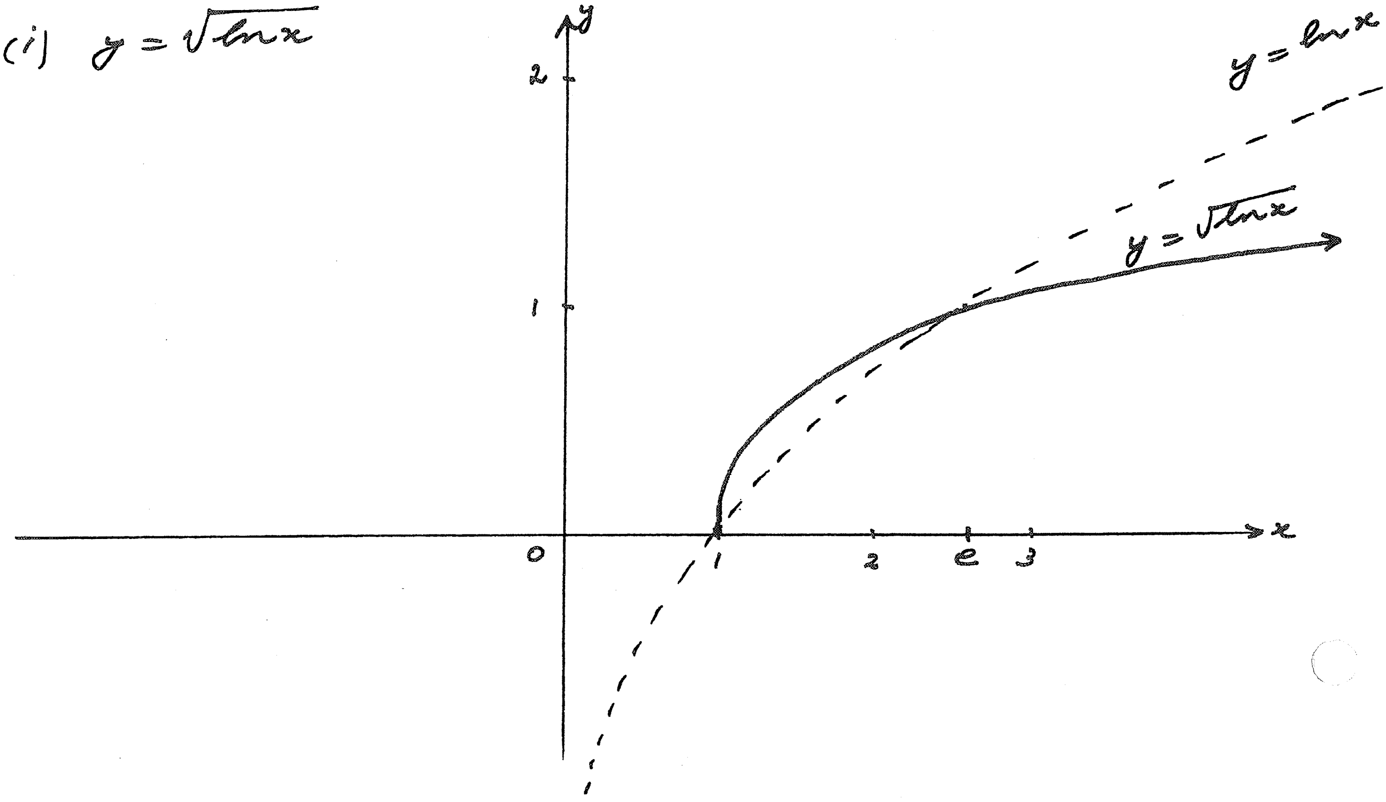
Area $\Delta OLM = \frac{1}{2}\cdot OL\cdot d$

$$= \frac{1}{2}\cdot\sqrt{11}(\sec\theta + \tan\theta)\cdot\frac{4\sqrt{7}}{\sqrt{11}}(\sec\theta - \tan\theta)$$

$= 2\sqrt{7}$ units (ie independent of θ)

QUESTION 7

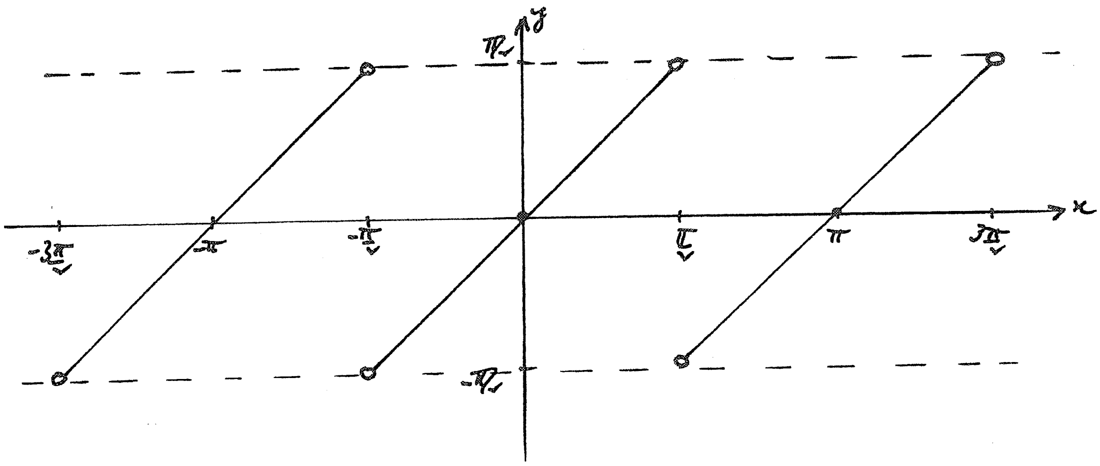
(a) (i) $y = \sqrt{\ln x}$



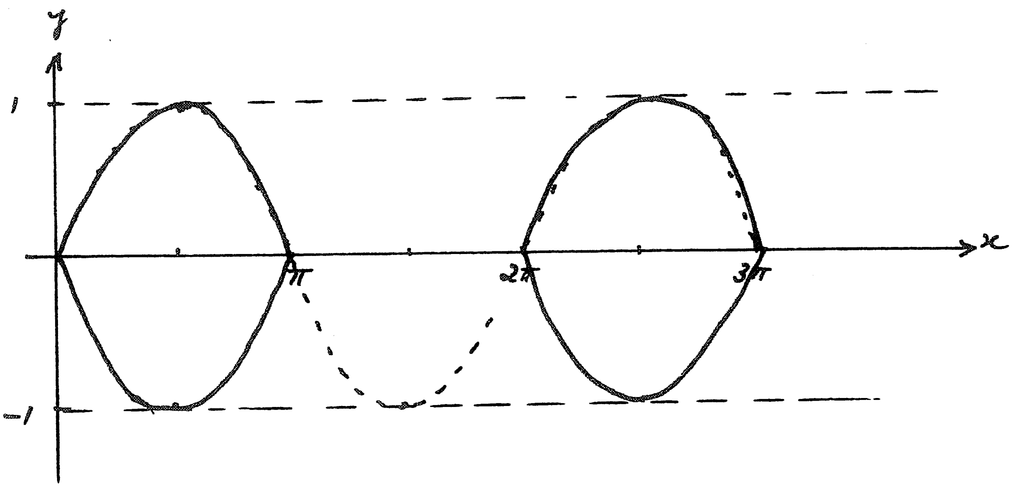
(ii) $y = \tan^{-1}(\tan x)$ for $-\frac{3\pi}{2} \le x \le \frac{3\pi}{2}$

$$\frac{dy}{dx} = \frac{1}{1+\tan^2 x} \cdot \sec^2 x$$

$$= 1 \text{ for all } x$$



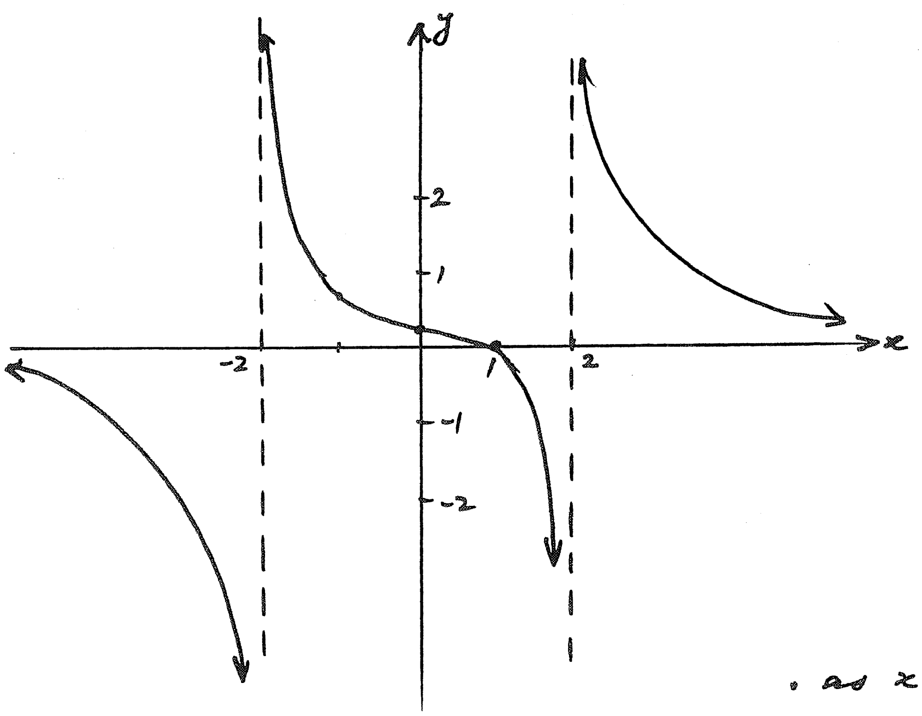
(iii)



$|y| = \sin x \Rightarrow \sin x \geq 0$

(iv) $y = \dfrac{x-1}{x^2-4}$

- x intercept at $(1,0)$
- y intercept at $(0, \frac{1}{4})$
- vert. asymp at $x = \pm 2$
- horiz. asymp at $y = 0$.
- $\lim_{x \to 2^+} \dfrac{x-1}{x^2-4}$ $+\infty$



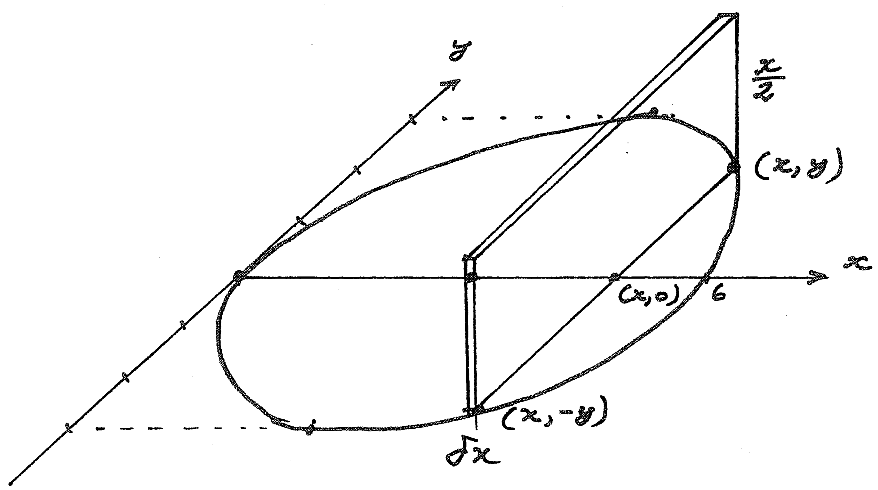
- as $x \to -2^-$, $y \to -\infty$

(b) $x^2 + y^2 = 6x$

$\Rightarrow x^2 - 6x + y^2 = 0$

$(x-3)^2 + y^2 = 9$

$\therefore$ Centre is $(3,0)$ radius is 3.



volume of slice shown is

$$\delta V = 2y \cdot \frac{x}{2} \cdot \delta x$$

$$= xy\,\delta x \quad \text{where } y = \sqrt{6x - x^2}$$

$\therefore$ Volume of solid is

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{6} xy\,\delta x$$

$$= \int_0^6 x\sqrt{6x - x^2}\,dx$$

$$6x - x^2 = -1(x^2 - 6x$$
$$= -1[(x-3)^2 -$$
$$= 9 - (x-3)^2$$

$$= \int_0^6 x\sqrt{9 - (x-3)^2}\,dx$$

let $x - 3 = 3\sin\theta$

$dx = 3\cos\theta\,d$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (3 + 3\sin\theta) \cdot \sqrt{9 - 9\sin^2\theta} \cdot 3\cos\theta\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (27\cos^2\theta + 27\underbrace{\sin\theta\cos^2\theta}_{\text{odd function} \therefore \text{zero}})\, d\theta$$

$$= 54\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta \qquad \cos 2\theta = 2\cos^2\theta - 1$$

$$= \frac{54}{2}\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)\, d\theta$$

$$= 27\left[\theta + \frac{1}{2}\sin 2\theta\right]$$

$$= 27\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{27\pi}{2}$$

$\therefore$ Volume is $\frac{27\pi}{2}$ units^3.

QUESTION 8:

(a) $f(\theta) = \cos\theta - \dfrac{1}{4\sqrt{3}\sin\theta}$

(i) $f'(\theta) = -\sin\theta - \left[\dfrac{-1.\,4\sqrt{3}\cos\theta}{(4\sqrt{3})^2\sin^2\theta}\right]$

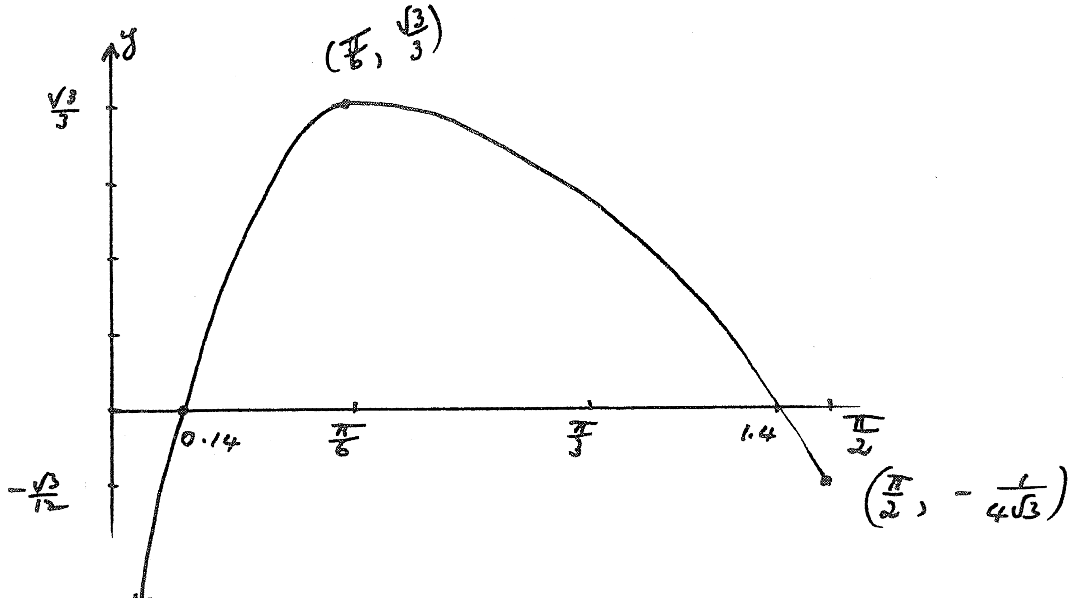
$= -\sin\theta + \dfrac{\cos\theta}{4\sqrt{3}\sin^2\theta}$

$\therefore f'\left(\frac{\pi}{6}\right) = -\frac{1}{2} + \dfrac{\frac{\sqrt{3}}{2}}{4\sqrt{3}.\,\frac{1}{4}}$

$= -\frac{1}{2} + \dfrac{\frac{\sqrt{3}}{2}}{\sqrt{3}}$

$= 0$

(ii)



as $\theta \to 0 \quad f(\theta) \to -\infty$

$f\left(\frac{\pi}{2}\right) = -\dfrac{1}{4\sqrt{3}}$

$f\left(\frac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2} - \dfrac{1}{2\sqrt{3}}$

$= \sqrt{3}$

max. turning point at $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{3}\right)$ since $f''(\theta) < 0$

x intercept at $y = 0$

ie $\cos\theta - \dfrac{1}{4\sqrt{3}\sin\theta} = 0$

ie $4\sqrt{3}\sin\theta\cos\theta = 1$

$2\sqrt{3}\sin 2\theta = 1$

$\sin 2\theta = \dfrac{1}{2\sqrt{3}}$

$2\theta = 0.29, 2.85$

$\theta \doteqdot 0.14, 1.4$

(b) (i) $x = Vt\cos\alpha$ ——— ①

$y = -\frac{1}{2}gt^2 + Vt\sin\alpha$ ——— ②

$y = 0 \Rightarrow -\frac{1}{2}gt^2 + Vt\sin\alpha = 0$

$-\frac{1}{2}t(gt - 2V\sin\alpha) = 0$

$\therefore t = \dfrac{2V\sin\alpha}{g}$

$\therefore$ Hits ground at $\dfrac{2V\sin\alpha}{g}$ s.

(ii) Range is $x = Vt\cos\alpha$

$= V\cos\alpha \cdot \dfrac{2V\sin\alpha}{g}$

$= \dfrac{2V^2\sin\alpha\cos\alpha}{g}$ m

(iii) Position of target is given by

$x = d + ut$ ——— ③

When projectile hits the target we have $t = \dfrac{2V\sin\alpha}{g}$

and $x = \dfrac{2V^2\sin\alpha\cos\alpha}{}$ $\Big\}$ sub in ③

$$\therefore \frac{2V^2\sin\alpha\cos\alpha}{g} = d + u\cdot\left(\frac{2V\sin\alpha}{g}\right)$$

ie $$u\cdot\frac{2V\sin\alpha}{g} = \frac{2V^2\sin\alpha\cos\alpha}{g} - d$$

$$\Rightarrow \quad u = V\cos\alpha - \frac{gd}{2V\sin\alpha} \quad \text{——} \; (4)$$

iv) We now assume that $gd = \dfrac{V^2}{2\sqrt{3}}$

Then (4) becomes

$$u = V\cos\alpha - \frac{V^2}{4V\cdot\sqrt{3}\sin\alpha}$$

$$= V\left[\cos\alpha - \frac{1}{4\sqrt{3}\sin\alpha}\right]$$ ie the same function as in part (a)

ie $$\frac{u}{V} = \cos\alpha - \frac{1}{4\sqrt{3}\sin\alpha}$$

now if $u > \dfrac{V}{\sqrt{3}}$

ie $\dfrac{u}{V} > \dfrac{1}{\sqrt{3}}$

ie $\dfrac{u}{V} > \dfrac{\sqrt{3}}{3}$

and from the graph in part (a) we see that no α gives a value of $\frac{u}{V}$ bigger than $\frac{\sqrt{3}}{3}$.

(v) If $u < \dfrac{V}{\sqrt{3}}$

ie $\dfrac{u}{V} < \dfrac{1}{\sqrt{3}}$ we see from the graph that two α give a solution ($u, V, > 0$)